Discrete Applied Mathematics 15 (1986) 329-364
North-Holland

A DECOMPOSITION OF THE MATROIDS WITH THE MAX-FLOW MIN-CUT PROPERTY

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Received 16 April 1986

A matroid has the max-flow min-cut property if a certain circuit packing problem has an optimal solution that is integral whenever the capacities assigned to the elements of the matroid are integral. P.D. Seymour characterized the matroids with this property in terms of minimal forbidden minors. Here we employ a variant of a previously developed decomposition algorithm to produce two decomposition theorems for this matroid class. The first theorem roughly says that any 3-connected matroid of the class is regular, or equal to the Fano matroid, or is a 3-sum. The second theorem is quite similar, but involves a more detailed analysis of the 3-sum case and includes an additional case.

1. Introduction

Suppose one selects an element l from a given matroid M with groundset S and constructs the following $\{0,1\}$ matrix H. The rows of H correspond to the elements of $S-\{l\}$, and the columns correspond to the circuits of M containing l. The entry of H in column C and row e is then 1 if element e occurs in circuit C, and 0 otherwise. Consider now the linear program

$$P(M,l)\begin{cases}\max\limits_{v}\ \mathbf{1}\cdot v,\\ \text{s.t.}\ \ Hv\le h,\\ \qquad\quad v\ge 0\end{cases}$$

where $\mathbf{1}$ is a vector of 1's. If M is the polygon matroid of an undirected graph with edge set S, then h_e may be viewed as the capacity of edge e, and $P(M,l)$ is the circulation problem demanding maximum flow on edge l. It is well known that this flow has an optimal solution with v integral provided h is nonnegative and integral, and that the dual of $P(M,l)$ has then an optimal integral solution as well [3]. These facts are neatly summarized by the max-flow min-cut theorem of network flows. An obvious question is then: Are there are other matroids M with an element l such that $P(M,l)$ has an integer optimal v whenever h is nonnegative and integral? When this

* Research of the second author was supported in part by the National Science Foundation under Grant No. MCS-8305462.

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requirement is satisfied by some M and l, we will say that the circuits of M with l *pack*, or simply that $P(M,l)$ *packs*. In that case the polyhedron of the dual of $P(M,l)$ has only integer extreme points [2] (indeed, one can easily prove that $u \cdot H - s = 1$, $u \geq 0$, $s \geq 0$ has the (stronger) property of 'local unimodularity' defined in [8]), and thus one has an attractive generalization of the max-flow min-cut theorem of network flows.

It is easily seen that the above question has an affirmative answer, so one next would ask for a characterization of the matroids with an element l for which $P(M,l)$ packs. Elementary arguments show that $P(\bar{M},l)$ packs for any minor $\bar{M}$ of M provided $P(M,l)$ packs ($\bar{M}$ must contain l, of course), and that $P(M,l)$ always packs if M has at most three elements. One is thus led to the following question: What are the minimal matroids M with an element l for which $P(M,l)$ does not pack? Seymour [5] provided the answer: Such minimal M must be isomorphic to U_4^2, the uniform matroid of rank 2 on four elements, or to F_7^*, the dual of the Fano matroid. If M is connected, we may rephrase this answer as follows: $P(M,l)$ packs if and only if M is binary (i.e., is representable over GF(2)) and no F_7^* minor of M contains l. While the part about U_4^2 is almost self-evident, Seymour's proof of the second requirement concerning F_7^* minors is long and complex.

In this paper we develop a decomposition for the matroids of the class $\mathcal{M} = \{M \mid M$ has an element l, and $P(M,l)$ packs$\}$, and also sketch a polynomial algorithm for finding the decomposition. The developments heavily rely on recent work on matroid connectivity and decomposition [11-13], and once more confirm the fundamental notion that decomposition can be very helpful when one looks for minimal violation matroids of properties inherited under the taking of minor, or attempts solution of complicated combinatorial problems.

The remaining sections of this paper are organized as follows. Section 2 contains some theorems on 3-connected extension sequences that will be invoked repeatedly. In Section 3 we describe a slightly modified version of the decomposition algorithm of [13] which we then use in Section 4 to obtain a basic decomposition theorem for the matroids of $\mathcal{M}$. Repeated application of the latter result in Section 5 produces a second decomposition theorem for the matroids of $\mathcal{M}$.

This paper is based on the Ph.D. dissertation of the first author [7], which was supervised by the second author. The reader will notice that the paper leaves open some questions, in particular those concerning the existence of a polynomial testing algorithm for membership in $\mathcal{M}$, or of a polynomial algorithm to solve $P(M,l)$, $M \in \mathcal{M}$. Very recently affirmative answers for a number of these questions, including the just mentioned ones, were found; they are described in [14], which also includes a proof of a strengthened form of Seymour's characterization.

Before going on, we would like to encourage the reader to go over Section 1 of [11] and over Sections 10 and 11 of [13] since we will make extensive use of that material, and since we will not repeat the intuitive reasoning included in [11, 13] for definitions and algorithms.

2. 3-connected extension sequences

This section is concerned with 3-connected extensions of matroids containing a certain element. Before we address this topic we introduce a few definitions about matroid representations. Let X be a base of a matroid M on a set S, and $Y=S-X$. Construct a $\{0,1\}$ matrix $\hat{B}=[I\,|\,B]$ as follows. S is to be the set of indices of the columns of $\hat{B}$; in particular X is to index the columns of the identity I, say in the order of $x_1, x_2, \ldots, x_m$. Then we index the rows by $x_1, x_2, \ldots, x_m$ as well. Let $y\in Y$, and suppose $\bar{X}$ is the subset of X that forms a circuit with y. Then in the column of B with index y we set element B_{xy} equal to 1 if $x\in\bar{X}$, and equal to 0 otherwise. Any $\hat{B}$ that may be so constructed from M is a *partial representation* of M, and it is nothing but a matrix representation of the fundamental circuit set of Whitney [18]. Note that this construction can always be carried out unless M consists only of loops. In the latter case we may formally take $\hat{B}$ to be a matrix without rows (we call a matrix without rows or columns *empty*).

Partial representations have been utilized by one of the authors in prior work [10–13], and the arguments to follow rely on the matrix theory for such representations developed in [10]. The definitions are motivated by the well-known relationship between the bases of a representable matroid and a related standard representation matrix $\hat{A}=[I\,|\,A]$. That is, the bases of the matroid are in one-to-one correspondence to the nonsingular submatrices of A save for the base corresponding to the submatrix I of $\hat{A}$. Thus we define for any square submatrix $\bar{B}$ of B, say specified by $\bar{X}\subseteq X$ and $\bar{Y}\subseteq Y$, a *determinant*, $\det\bar{B}$, which is declared to be 1 if $(X-\bar{X})\cup\bar{Y}$ is a base of M, and to be 0 otherwise. This definition is extended to square submatrices $\bar{B}$ of $\hat{B}$, say specified by $\bar{X}\subseteq X$ and $\bar{Z}\subseteq X\cup Y$, by defining $\det\bar{B}$ to be equal to 1 if the set $(X-\bar{X})\cup\bar{Z}$ is a base of M, and to be equal to 0 otherwise. There may be another submatrix of $\hat{B}$ that (possibly after row and/or column permutations) is numerically identical to $\bar{B}$. The related row and column index sets are not both identical to $\bar{X}$ and $\bar{Z}$, respectively, and for this reason we will consider such a matrix to be different from $\bar{B}$. Thus for mathematical exactness we could specify $\bar{B}$ of $\hat{B}$ by the triple $(\bar{X},\bar{Z},X)$. We avoid this cumbersome notation since confusion of $\bar{B}$ with some other matrix seems unlikely. We also use expressions like "$\bar{B}$ is singular (nonsingular)" with the obvious interpretation. If $\bar{B}$ is a (not necessarily square) submatrix of $\hat{B}$, we define $\operatorname{rank}(\bar{B})$ to be the order of the largest nonsingular submatrix of $\bar{B}$. If $\bar{B}$ is indexed by $\bar{X}$ and $\bar{Z}$ as before, and if $r(\cdot)$ is the rank function of M, then it is easily verified that $\operatorname{rank}(\bar{B})=r((X-\bar{X})\cup\bar{Z})-|X-\bar{X}|$. We say "$B^1$ spans C" if B^1 has the same rank as $[B^1\,|\,C]$ or $[\frac{B^1}{C}]$ whichever applies. If M is binary (i.e., representable over GF(2)), then the determinant of any $\bar{B}$ is nothing but $\det\bar{B}$, viewing $\bar{B}$ as a matrix over GF(2). $\hat{B}$ is then called a *standard representation* of M over GF(2). Though this paper is mainly concerned with binary matroids, we have chosen to present the results of this section and the next one in terms of general (not just binary) matroids since they likely will be of use elsewhere.

A *pivot* on element $\hat{B}_{xy}=1$ of $\hat{B}$ transforms $\hat{B}$ into the partial representation corresponding to the base $(X-\{x\})\cup\{y\}$. For the display of the new partial representation we find it convenient to switch the position of the column indices of x and y, and to leave the position of all other column indices unchanged. We may evaluate determinants by pivots as follows. Let a square submatrix $\bar{B}$ of $\hat{B}$ of order at least 2 be indexed by $\bar{X}$ and $\bar{Z}$ as before. If we pivot on the (x,y) element of $\bar{B}$, then in the resulting partial representation the submatrix indexed by $\bar{X}-\{x\}$ and $\bar{Z}-\{y\}$ has the same determinant as $\bar{B}$.

Two nonzero rows (columns) of a $\bar{B}$ are *parallel* if the submatrix consisting of these rows (columns) has rank equal to 1. Note that a column unit vector of B is always parallel to the related unit vector of the identity I in $\hat{B}$. The matrix $[B^t\,|\,I]$ is a partial representation of M^*, the dual of M, and any square submatrix $\bar{B}$ of B is nonsingular if and only if $(\bar{B})^t$ is nonsingular in B^t (in the triple notation mentioned above $(\bar{B})^t$ becomes $(\bar{Z},\bar{X},Y)$). If we delete a column with index $y\in Y$ from $\hat{B}$, we obtain a partial representation of $M\setminus y$, where '$\setminus$' denotes deletion. (If e is an element, we write $M\setminus e$ and M/e instead of $M\setminus\{e\}$ and $M/\{e\}$ to unclutter the notation). The determinants of the square submatrices of the reduced matrix are unchanged by such a column deletion. By duality a deletion of a row and column of $\hat{B}$ with index $x\in X$ produces a partial representation of M/x, where '/' denotes contraction. Again, the determinants are not affected by this operation. An *addition* (*expansion*) is the inverse of a deletion (contraction). An *extension* is an addition or an expansion. Note that the definition of *contraction* differs from that by Tutte (see, e.g., [16]), and that another definition of *extension* is given by Welsh [17, p. 319].

Let M be a matroid with rank function $r(\cdot)$ on a set S. If two elements of S form a circuit in M (M^*), they are said to be *parallel* (*series*) elements. Any circuit of cardinality equal to 3 in M (M^*) is a *triangle* (*triad*). M is *k-separable* [16] if S can be partitioned into S_1 and S_2 such that $|S_1|,\ |S_2|\geq k$ and $r(S_1)+r(S_2)\leq r(S)+k-1$. The pair (S_1,S_2) is then a *k-separation* of M, which manifests itself in the previously defined $\hat{B}$ as follows. Let $X_i=X\cap S_i$ and $Y_i=Y\cap S_i$, $i=1,2$. If we partition B as

$$B=\begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & B^{11} & B^{12} \\ \hline X_2 & B^{21} & B^{22} \\ \hline \end{array}$$

then by the previously mentioned relationship between rank$(\cdot)$ and $r(\cdot)$ we have $\mathrm{rank}(B^{12})=r(X_2\cup Y_2)-|X_2|$ and $\mathrm{rank}(B^{21})=r(X_1\cup Y_1)-|X_1|$, and therefore $r(S_1)+r(S_2)\leq r(S)+k-1$ if and only if $\mathrm{rank}(B^{21})+\mathrm{rank}(B^{12})\leq k-1$. M is *k-connected* if it has no j-separation, $j\leq k-1$; a 2-connected M is also said to be *connected*. For a given $k\geq 2$, M is *(k+)-separable* if

(1) M is $(\lceil k/2\rceil+1)$-connected,

(2) both the rank and the corank of M are at least k, and

(3) M has a k-separation where the sets S_1 and S_2 satisfy $|S_1|$, $|S_2| \geq k+1$.
Here $\lceil n \rceil$ denotes the smallest integer greater than or equal to n. The pair (S_1, S_2) is then a *$(k+)$-separation* of M. A k-separation or $(k+)$-separation (S_1, S_2) is *exact* if $r(S_1)+r(S_2)=r(S)+k-1$.

Let A be a matrix. Then $\hat{A}$ denotes $[I|A]$, where I is an identity of appropriate order. In the display of matrices unspecified entries are always to be taken as 0. We typically write the index sets of the columns above a matrix and those of the rows to the left of it, and for $\hat{A}$ the column indices of I are always the same as the row indices of $\hat{A}$. If A has size $m \times n$, the the *length* of A is 0 if m or n is 0, and is $m+n$ otherwise. We define $G(A)$ to be the following bipartite graph. Each row and each column of A generates a node, and each nonzero A_{ij} leads to an edge connecting nodes i and j. We say that A is *connected* if $G(A)$ is connected. Partial representations allow a simple characterization of matroid connectivity as follows.

Lemma 2.1 (Cunningham [1], Krogddahl [4]). *Let $\hat{B}$ be a partial representation of a matroid M. Then M is connected if and only if $\hat{B}$ is connected.*

Given two matroids M and N both of whose groundsets contain a set L, and further given a set of functions $F \subseteq \{f$ is a bijection from L to $L\}$ that includes the identity function if $L \neq \emptyset$, and that is closed under the taking of inverse, we say that M is *(L,F)-isomorphic* to N, denoted by $M \cong_{(L,F)} N$, if there exists an isomorphism between M and N that maps L in M onto L in N according to some $f \in F$. If L contains just one element, say l, we simply use l-isomorphism and $\cong_l$. For isomorphism this notation is further reduced to $\cong$. We also use *(L,F)-automorphism* and *l-automorphism* with analogous interpretation.

Certain matroids are of particular interest. U_4^2 denotes the uniform matroid of rank 2 on four elements, and F_7 is the Fano matroid, with binary standard representation $\hat{B}$ where

$$B=\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The *wheel* W_m (whirl $\mathcal{W}_m$), $m \geq 3$, is the matroid on $2m$ elements with a partial representation $\hat{B}$ where

$$\begin{array}{c} Y \\ B=\begin{bmatrix} 1 & & & & & 1 \\ 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & \ddots & & \\ & & & \ddots & 1 & \\ & & & & 1 & 1 \end{bmatrix} \end{array}$$

and Y is dependent (independent). It is convenient for us to also consider U_4^2 to be a whirl, denoted by $\mathcal{W}_2$. Any other matroid terminology not covered here may be found in the book by Welsh [17].

We are now ready to introduce and prove the main theorem of this section.

Theorem 2.2. *Let M be a 3-connected matroid with a 3-connected proper minor N on four or more elements, among them one element labelled l. Then at least one of the statements below applies.*

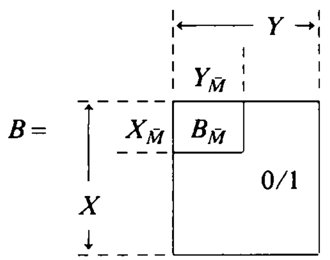
(i) *M has a minor $N_1 \cong_l N$ and another 3-connected minor $\bar{M}$ that is a 1-element extension of N_1.*

(ii) *$N = \mathcal{W}_m$, some $m \geq 2$. Further, M has a minor $N_1 \cong_l N$ and a second minor $\bar{M} \cong \mathcal{W}_{m+1}$ that is a 2-element extension of N_1.*

(iii) *$N \cong W_m$, some $m \geq 3$. Further, M has a minor $N_1 \cong_l N$ and a second minor $\bar{M} \cong W_{m+1}$ that is a 2-element extension of N_1.*

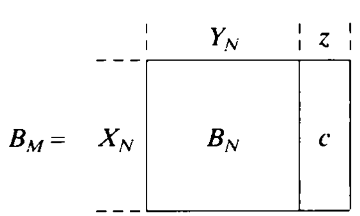
The proof of Theorem 2.2 relies on the following result.

Theorem 2.3 [10]. *Let M be a 3-connected matroid with a 3-connected proper minor N on four or more elements. Then M has a 3-connected minor $\bar{M}$ that has N as a proper minor, and that has at most three elements beyond those of N. Specifically, M has a partial representation $\hat{B}$ where*



Here $B_{\bar{M}}$ corresponds to $\bar{M}$ (so $\bar{M} = M/(X - X_{\bar{M}}) \setminus (Y - Y_{\bar{M}})$), and it is one of the matrices below, where in each case B_N corresponds to the minor N, and x, y, and z are the additional elements $\bar{M}$ has beyond those of N.

(2.4)



c is not a zero or unit vector, and it is not parallel to a column of B_N.

(2.5)
$$B_M = \begin{array}{c|c|} & Y_N \\ \hline X_N & B_N \\ \hline z & c \\ \hline \end{array}$$

c is not a zero or unit vector, and it is not parallel to a row of B_N.

(2.6)
$$B_{\bar{M}} = \begin{array}{c|c|c|} & Y_N & y \\ \hline X_N & B_N & c \\ \hline x & d & \alpha \\ \hline \end{array}$$

(i) *c (d) is a unit vector, or it is parallel to a column (row) of B_N.*
(ii) *Each of $[\frac{c}{\alpha}]$, $[d\,|\,\alpha]$ is not a unit vector, and it is not parallel to a column/row of the remainder of $B_{\bar{M}}$.*
(iii) *If c (d) is parallel to the j-th column (row) of B_N, then d (c) is not the j-th unit vector.*

(2.7)
$$B_{\bar{M}} = \begin{array}{c|c|c|c|} & Y_N & y & z \\ \hline X_N & B_N & c & e \\ \hline x & 0 & 1 & 1 \\ \hline \end{array}$$

(i) *Each of c, e is a unit vector or parallel to a column of B_N.*
(ii) *c is not parallel to e.*

(2.8)
$$B_{\bar{M}} = \begin{array}{c|c|c|} & Y_N & y \\ \hline X_N & B_N & 0 \\ \hline x & c & 1 \\ \hline z & e & 1 \\ \hline \end{array}$$

(i) *Each of c, e is a unit vector or parallel to a row of B_N.*
(ii) *c is not parallel to e.*

Proof of Theorem 2.2. By Theorem 2.3, M has a 3-connected minor $\bar{M}$ with a partial representation $\hat{B}_{\bar{M}}$, where $B_{\bar{M}}$ is one of the matrices of (2.4)–(2.8). In case of (2.4) or (2.5) we are done. For (2.7) and (2.8) we may suppose that both vectors c and e are parallel to distinct columns or rows of B_N. If this is not so, pivots in B_N can produce this configuration since, e.g., c of (2.7) must be a unit vector if it is not parallel to a column of B_N. Consider (2.7). One vector of c and e, say e, is parallel to a column $u \neq l$ of B_N. Delete column u from $B_{\bar{M}}$ and put column z (which contains e) in its place. This changes B_N to, say, B_{N_1} of a minor $N_1 \cong_l N$. If we declare N_1 to be N (which we may do for our purposes since l-isomorphism is an equivalence relation), we now have case (2.6). The analogous reduction is possible for (2.8), so only (2.6) requires further analysis. For simplicity we denote $B_{\bar{M}}$ of that case from now on by B, so $\hat{B}$ is a partial representation of a 2-element 3-connected extension of N.

Due to pivots we may assume that the $m \times n$ matrix B is actually

$$
(2.9) \qquad B = \begin{array}{c|c|c|}
 & Y_N & y \\
\hline
v & e & 1 \\
\hline
X_N & \bar{B} & 0 \\
\hline
x & d & 1 \\
\hline
\end{array}
$$

where e is not parallel to d. In each partial representation $\hat{B}$ of $\bar{M}$ displayed below, the $(m-1) \times (n-1)$ submatrix in the upper left corner is always called B_N, and it corresponds to a minor l-isomorphic to N. When we use the same letter for two vectors we imply that they are parallel. With these conventions we see that the last column (row) of any B after deletion of the m-th (n-th) element must be a unit vector or it must be parallel to a column (row) of B_N, or we have found an instance of (i) of Theorem 2.2 and can stop. Thus we may assume that the former situation always prevails. Also note that m and n must be at least 3. We are now ready for the proof.

Initially we assume that v, the index of the first row of B, is actually l. If d is a unit vector, we can convert it to a non-unit vector by a pivot in $\bar{B}$ without changing column y. Such a pivot leaves l as the index of the first row of the new B. If such a pivot is not possible, $\bar{M}$ is clearly 2-separable, a contradiction. So let d be parallel to the first row of $\bar{B}$, say with index $\bar{x}$. An exchange of rows x and $\bar{x}$ converts the second element in column y to a 1, and replaces N by an l-isomorphic minor. Column y minus the last element must be parallel to a column of B_N, say the first column, with index $\bar{y}$ .An exchange of columns y and $\bar{y}$ results in

$$
(2.10) \qquad B = \begin{array}{c|c|c|c|}
 & y & & \bar{y} \\
\hline
v & 1 & \bar{e} & 1 \\
\hline
x & 1 & \bar{d} & 1 \\
\hline
 & 0 & 0/1 & 0 \\
\hline
\bar{x} & 0 & \bar{d} & 1 \\
\hline
\end{array}
$$

The subvectors $[\bar{d}\,|\,1]$ of rows x and $\bar{x}$ in (2.10) are the d's of (2.9), so they are parallel. The 2×2 submatrix specified by indices $v, x, y, \bar{y}$ is singular, so a pivot on B_{xy} changes the first element in column $\bar{y}$ to a 0 and results in

$$
\text{(2.11)} \qquad B = \begin{array}{c|c|c|c|}
 & x & & \bar{y} \\ \hline
v & 1 & \bar{e} & 0 \\ \hline
y & 1 & \bar{d} & 1 \\ \hline
 & 0 & 0/1 & 0 \\ \hline
\bar{x} & 0 & \bar{d} & 1 \\ \hline
\end{array}
$$

By (2.11) and induction we thus may assume that B is of the form

$$
\text{(2.12)} \qquad B = \begin{array}{cc|c|c|c|}
 & & Y_1 & Y_2 & y \\ \hline
\multirow{4}{*}{X_1} & v & \begin{matrix} 1 & & & \\ 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{matrix} & \begin{matrix} e \\ \hline 0 \\ \hline d \end{matrix} & \begin{matrix} 0 \\ \hline 0 \\ \hline 1 \end{matrix} \\ \hline
 & X_2 & 0 & 0/1 & 0 \\ \hline
 & x & 0 & d & 1 \\ \hline
\end{array}
$$

where the two subvectors of type $[d\,|\,1]$ (with column index set $Y_2 \cup \{y\}$) are parallel. If d is not a unit vector, then it is parallel to a row of B_N with index in X_2. We then do the same arguments as before to produce an instance of (2.12) where $|Y_1|$ is increased by 1. If d is a unit vector, say with the 1 in column $\bar{y} \in Y_2$, and if the vector defined by X_2 and column $\bar{y}$ is not zero, then we pivot on a 1 in that vector and produce another instance of (2.12) where $|Y_1|$ is as before, but d has become a non-unit vector. If this pivot is not possible, Y_2 must be $\{\bar{y}\}$ and $X_2 = \emptyset$ since otherwise $\bar{M}$ is 2-separable. By induction we may thus assume $Y_2 = \{\bar{y}\}$ and $X_2 = \emptyset$. The first element of column $\bar{y}$ must be a 1 since otherwise row 1 of B is a unit vector. We conclude that $\bar{M}$ is isomorphic to a wheel or whirl. A pivot on B_{xy} then shows $\bar{M}$ to be isomorphic to a wheel if N is, and to a whirl otherwise.

So far we have presumed that $v = l$, so we now remove this restriction. The above proof still applies, but we now must make sure that any column or row exchanges will not involve a column or row of B_N with index l. In all cases of row exchanges involving l B may be re-arranged to

$$B = \begin{array}{c} \\ l \\ \\ x \end{array} \begin{array}{|c|c|} \hline e & 1 \\ \hline d & \\ \cline{1-1} 0/1 & 0 \\ \hline d & -1 \\ \hline \end{array}$$

For a proof just examine (2.9) and (2.12). Pivot in row l of B and pass to the dual. Upon a change of notation of index sets (except for l) and matrices we then have (2.9) where the submatrices e and $\bar{B}$ represent the dual of N, and v is actually l. In all cases of column exchanges involving l B may be re-arranged to

$$B = \begin{array}{ccc} l & & y \\ \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 0 \\ \hline 1 \\ \hline \end{array} & \begin{array}{|c|} \hline e \\ \hline d \\ \hline 0/1 \\ \hline d \\ \hline \end{array} & \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \end{array}$$

where the two row subvectors of type $[1 \mid d]$ are parallel, and the determinant of the 2×2 submatrix defined by the first and second row and the columns with index l and y, is 0. Pivot on the 1 in the upper left corner. Upon a change of notation of index sets (except for l) and matrices we again have (2.9) with $v = l$.

Thus by the above proof N is indeed isomorphic to a wheel or whirl. □

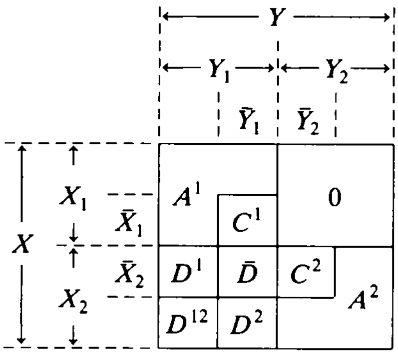
One additional simple 3-connectivity result will be of use.

Lemma 2.13 [13]. *Let a connected matroid M have a 3-connected minor N with at least three elements, and l be an element in M. Then M has a minor N_1 containing l that is either isomorphic to N, or that is a 3-connected 1-element extension of N.*

3. A decomposition algorithm for matroids with special subsets

In this section we extend the recursive algorithm for the construction of decomposition theorems of [13] so that matroid classes whose matroids contain special subsets, for example $\mathcal{M}$ defined in the Introduction, can be processed.

We begin with a few definitions taken from [11]. A matroid M is a *1-sum* if it is the disjoint union of two matroids M_1 and M_2. This situation is denoted by $M = M_1 \oplus_1 M_2$. M is a *k-sum*, $k \geq 2$, if M has a partial representation $\hat{B}$ where

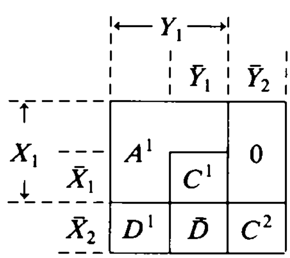
(3.1) $B=$ 

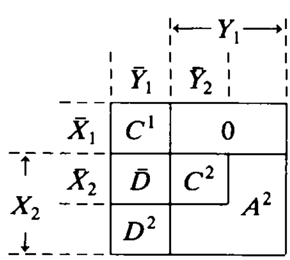
observes the following conditions.

(3.2)
(a) C^1 (C^2) is a connected nonempty proper submatrix of A^1 (A^2), and it has no nested rows (columns).
(b) $\bar{D}$ is a nonsingular matrix and $\operatorname{rank}(\bar{D}) = \operatorname{rank}(D) = k-1$, where

$$D = \begin{array}{|c|c|} \hline D^1 & \bar{D} \\ \hline D^{12} & D^2 \\ \hline \end{array}$$

We recall that two $\{0,1\}$ vectors c and d are *nested* if $c_j = 1$ implies $d_j = 1$, $\forall j$, or $d_j = 1$ implies $c_j = 1$, $\forall j$. By the previous observations the matrices $\hat{B}^1$ and $\hat{B}^2$ defined by

(3.3) $B^1 =$ 

$B^2 =$ 

are partial representations of $M_1 = M/(X_2 - \bar{X}_2) \setminus (Y_2 - \bar{Y}_2)$ and $M_2 = M/(X_1 - \bar{X}_1) \setminus (Y_1 - \bar{Y}_1)$ respectively, and the determinants of their submatrices agree with those of $\hat{B}$. We call M_1 and M_2 the *components* of the k-sums and write $M = M_1 \oplus_k M_2$. Note that $\oplus_k$ is not commutative, for all $k \geq 2$. It is trivial to verify that $M = M_1 \oplus_k M_2$ if and only if $M^* = M_2^* \oplus_k M_1^*$. Note that the minor

$$\bar{M} = M/((X_1 - \bar{X}_1) \cup (X_2 - \bar{X}_2)) \setminus ((Y_1 - \bar{Y}_1) \cup (Y_2 - \bar{Y}_2))$$

is present in both M_1 and M_2. For $k \geq 3$, $\bar{M}$ is always 3-connected, and a comparison of (3.1) and (3.3) reveals that identification (in some sense) of the $\bar{M}$ minor of M_1 with the $\bar{M}$ minor of M_2 produces M from M_1 and M_2. For this reason we call $\bar{M}$ the connecting matroid of the decomposition. Assume that M_1 and M_2 are representable over a given field, and that they have standard representation matrices $\hat{B}^1$ and $\hat{B}^2$ (not necessarily $\{0, 1\}$) whose B^1 and B^2 are give by (3.3) and satisfy (3.2) with $D^{12} = D^2 \bar{\bar{D}} D^1$, where $\bar{\bar{D}}$ is the inverse of $\bar{D}$. Then we can compose M_1 and M_2 to a matroid M (which we also denote by $M_1 \oplus_k M_2$) by defining M to be the matroid represented by $\hat{B}$ with B of (3.1) over that field, where the submatrices of B are those of (3.3) and D^{12} is the matrix just specified. If both M_1 and M_2 are representable over two or more fields, this procedure may generate several matroids depending on the field over which B^1 and B^2 are expressed. Though the notation "$M_1 \oplus_k M_2$" may thus be ambiguous when used for composition, this will not cause any difficulty here since the underlying field will always be GF(2). A k-sum is *proper* if both submatrices A^1 and A^2 of B of (3.1) are connected; it is *semi-proper* if one of the following two situations prevails: either A^1 is connected and A^2 is equal to C^2 with one additional zero row adjoined to the bottom of C^2, or A^2 is connected and A^1 is equal to C^1 with one additonal zero column adjoined to the left hand side of C^1.

Next we review and extend relevant decomposition results of [13]. For the remainder of this section $\mathcal{M}$ is a class of matroids whose groundsets always contain a given (possibly empty) set L. Also given is a set F of functions, $F \subseteq \{f \mid f$ is a bijection from L to $L\}$, that includes the identity function if $L \neq \emptyset$, and that is closed under the taking of inverse. If $M \in \mathcal{M}$, then all minors of M containing L as well as all matroids that are (L, F)-isomorphic to M, are also in $\mathcal{M}$. Now suppose that an $M \in \mathcal{M}$ with a groundset S has a minor $N = (M/X_3 \setminus Y_3) \in \mathcal{M}$ where X_3 contains no circuit, Y_3 contains no cocircuit, and $X_3 \cap Y_3 = \emptyset$. Suppose for some $k \geq 1$ we know of an exact k-separation (T_1, T_2) of $T = S - (X_3 \cup Y_3)$ of N. Below all k-separations and $(k+)$-separations are required to be exact, so as a matter of convenience we omit 'exact' throughout this section. We then may ask whether or not one can extend this k-separation to one for M, or more precisely, whether or not one can assign the elements of $S - T$ to T_1 and T_2 so that the resulting sets S_1 and S_2 constitute a k-separation (S_1, S_2) of M. We say "the k-separation of N induces a k-separation of M" if such an assignment is possible. The assumed k-separation may be viewed as follows. Choose a base $X_2 \subseteq T_2$, then add a set $X_1 \subseteq T_1$ so that $X_1 \cup X_2$ is a base of N. The partial representation $\hat{B}_N$ produced from this base has

$$(3.4)\qquad B_N = \begin{array}{c|c|c|} & Y_1 & Y_2 \\ \hline X_1 & A^1 & 0 \\ \hline X_2 & D & A^2 \\ \hline \end{array}\;; \qquad X_i \cup Y_i = T_i, \quad i = 1,2$$

with $\operatorname{rank}(D) = k-1$. To simplify the exposition, we assume throughout this section that $X_i, Y_i \neq \emptyset$, $i = 1,2$. If (T_1, T_2) is a $(k+)$-separation, then the latter assumption holds, or it becomes satisfied when the roles of T_1 and T_2 are reversed [11]. In any case the results described below are easily adapted to the situation where one or two of the four sets are empty. (The case with two empty sets is quite uninteresting.) Since $N = M/X_3 \setminus Y_3$, M has a partial representation $\hat{B}_M$ with

$$(3.5)\qquad B_M = \begin{array}{cc|c|c|c|c|} & & & \multicolumn{2}{c|}{\overleftrightarrow{\;Y_3\;}} & \\ & & Y_1 & Y_{31} & Y_{32} & Y_2 \\ \hline & X_1 & A^1 & & & 0 \\ \cline{2-2} \multirow{2}{*}{X_3} & X_{31} & & \tilde{A}^1 & 0 & \\ \cline{2-2} & X_{32} & & \tilde{D} & \tilde{A}^2 & \\ \cline{2-2} & X_2 & D & & & A^2 \\ \hline \end{array}$$

The submatrix composed of $A^1, 0, D, A^2$ has the same determinantal structure as B_N of (3.4). Then the k-separation of N induces one for M if and only if we can partition X_3 into X_{31}, X_{32} and Y_3 into Y_{31}, Y_{32} as shown in (3.5) such that

$$(3.6)\qquad \operatorname{rank}(\tilde{D}) = \operatorname{rank}(D) \quad (= k-1).$$

How can we detect such a partition of X_3 and Y_3, or show that none exists? Let $x \in X_3$ and $y \in Y_3$. Rewrite B_M of (3.5) as

$$(3.7)\qquad B_M = \begin{array}{c|c|c|c|c|} & Y_1 & y & Y_3 & Y_2 \\ \hline X_1 & A^1 & g & & 0 \\ \hline x & e & & & f \\ X_3 & & & & \\ \hline X_2 & D & h & & A^2 \\ \hline \end{array}$$

Examine the row of B_M indexed by $x \in X_3$. Suppose $\operatorname{rank}([\frac{e}{D}]) > \operatorname{rank}(D)$. Then x must be in X_{31} in any partition of B_M satisfying (3.5) and (3.6). If in addition $f \neq 0$, then x must also be in X_{32} by (3.5); i.e., B_M cannot be partitioned. On the other hand, suppose $f = 0$. Define N_1 to be the minor of M that corresponds to the submatrix of B_M composed of $A^1, D, 0, A^2, e, f$. Then N_1 has a k-separation $(X_1 \cup \{x\} \cup Y_1, X_2 \cup Y_2)$ that induces one for M if and only if N does. We thus have obtained a smaller problem since N_1 and M have more elements in common than N and M. In an algorithm one would want to view the derivation of N_1 from N as a *shifting* of x from X_3 to X_{31}.

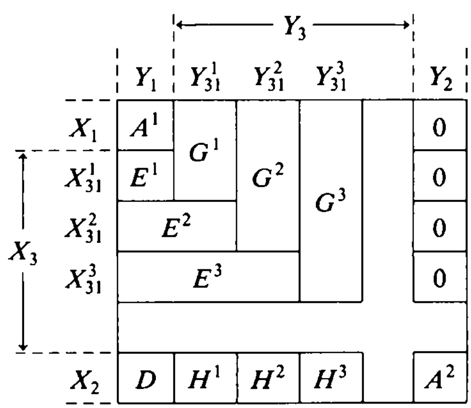
Quite similarly we can process column $y \in Y_3$ of B_M of (3.7). Suppose $g \neq 0$. Then M has no induced k-separation if $\mathrm{rank}([D \mid h]) > \mathrm{rank}(D)$, while equality in the latter expression implies that the k-separation $(X_1 \cup Y_1 \cup \{y\}, X_2 \cup Y_2)$ of the minor N_2 of M corresponding to the submatrix composed of $A^1, D, 0, A^2, g, h$ induces one for M if and only if this is so for N. Again, we have a smaller problem in the latter case, and the derivation of N_1 could be viewed to be a *shifting* of y from Y_3 to Y_{31}.

Finally suppose that g is zero for all columns $y \in Y_3$, and $\mathrm{rank}(\begin{bmatrix} e \\ D \end{bmatrix}) = \mathrm{rank}(D)$ for all rows $x \in X_3$. Then we may choose $X_{31} = \emptyset$, $X_{32} = X_3$, $Y_{31} = \emptyset$, and $Y_{32} = Y_3$, and thus have partitioned B_M as required by (3.5) and (3.6).

The above discussion clearly suggests a polynomial algorithm for deciding whether or not M has an induced decomposition. But we want to go further and understand the structure of M when no induced decomposition exists. For this reason we carry out the shifting of the elements of X_3 and Y_3 to X_{31} and Y_{31}, respectively, in the following order.

Initially $X_{31} = Y_{31} = \emptyset$. First we shift all rows that must be shifted, then columns, then rows again, etc. If during two successive shifting steps we do not shift any row or column, then the current X_{31}, Y_{31}, and $X_{32} = X_3 - X_{31}$, $Y_{32} = Y_3 - Y_{31}$ give the desired partition of B_M. We also stop when we detect that a just-shifted row must be in both X_{31} and X_{32}, or that a just-shifted column must be in both Y_{31} and Y_{32}; we then conclude that N does not induce a k-separation. Let us examine an instance of the latter case. Suppose according to the above process we first shift rows indexed by, say X_{31}^1, then columns indexed by Y_{31}^1, rows indexed by X_{31}^2, etc., until we shift columns of Y_{31}^3 and detect that N does not induce a k-separation. At that time we can partition B_M as follows.

(3.8)



The shifting stops with Y_{31}^3 because a column $y_3 \in Y_{31}^3$ of H^3 is not spanned by D.

In column y_3 of G^3 we must have a 1 since that column was shifted. Indeed, a 1 can only occur in a row $x_3 \in X_{31}^3$, since any 1 in a row of $X_1 \cup X_{31}^1 \cup X_{31}^2$ would have caused a shifting of y_3 when Y_{31}^1 or Y_{31}^2 was determined. Furthermore, $[D|H^1|H^2]$ does not span row x_3 of E^3, but $[D|H^1]$ does span the column submatrix of E^3 defined by $Y_1 \cup Y_{31}^1$, since otherwise row x_3 would have been shifted as part of X_{31}^1 or X_{31}^2. Thus we can pick an index $y_2 \in Y_{31}^2$ such that the column submatrix of $[D|H^1|H^2]$ indexed by $Z = Y_1 \cup Y_{31}^1 \cup \{y_2\}$ does not span the subvector of row x_2 of E^2 indexed by Z. In column y_2 we must find a 1 in a row indexed by some $x_2 \in X_{31}^2$, etc. Continued backtracking in the just-described fashion yields additional row and column indices, until we find a $y_1 \in Y_{31}^1$ such that vector y_1 of G^1 has a 1 in a row of X_1, or until we locate an $x_1 \in X_{31}^1$ such that D does not span row x_1 of E^1. The x_i and y_i so found may be employed to extract one of the following two matrices from B_M.

$$
B_M^1 = \begin{array}{c|c|ccc|c|}
 & Y_1 & y_1 & y_2 & y_3 & Y_2 \\
\hline
X_1 & A^1 & & & & 0 \\
\hline
x_1 & e & 1 & & & \\
x_2 & r^1 & \alpha_1 & 1 & 0 & \\
x_3 & \multicolumn{2}{c}{r^2} & \alpha_2 & 1 & \\
\hline
X_2 & D & c^1 & c^2 & h & A^2 \\
\hline
\end{array}
\quad ; \quad
B_M^2 = \begin{array}{c|c|ccc|c|}
 & Y_1 & y_1 & y_2 & y_3 & Y_2 \\
\hline
X_1 & A^1 & g & & & 0 \\
\hline
x_2 & r^1 & \alpha_1 & 1 & & 0 \\
x_3 & \multicolumn{2}{c}{r^2} & \alpha_2 & 1 & \\
\hline
X_2 & D & c^1 & c^2 & h & A^2 \\
\hline
\end{array}
$$

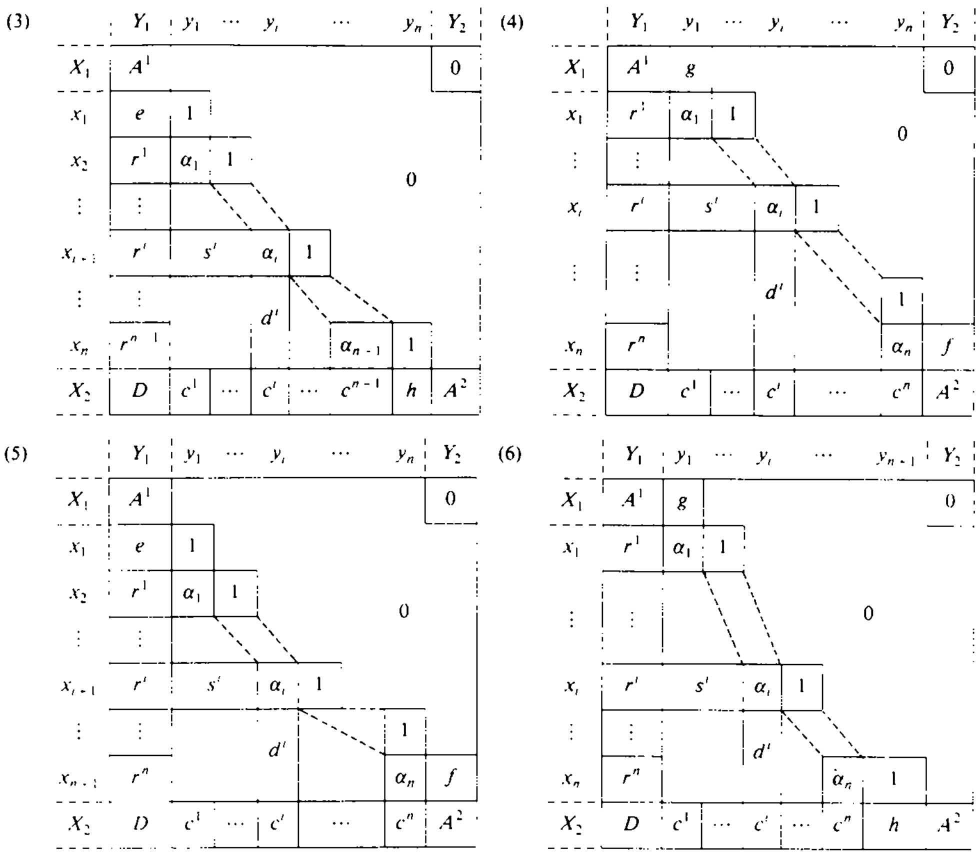
By the derivation of the x_i and y_i, we can claim the following results for B_M^1 and B_M^2. Vectors e and h are not spanned by D, and $g \neq 0$. Let D^i be the submatrix defined by the column index of any α_i and of all columns to the left of α_i, and by the row index of α_i and of all rows below α_i. Then $\text{rank}(D^i) = \text{rank}(D) + 1$, while $\text{rank}(\bar{D}) \leq \text{rank}(D)$ for any submatrix $\bar{D}$ of D^i not containing α_i.

The above discussion is rather easily extended to a proof for the following theorem.

Theorem 3.9 [13]. *B_M of* (3.5) *cannot be partitioned such that* (3.6) *holds if and only if one of the six matrices below may be derived from B_M by deletion of some rows of X_3 and some columns of Y_3. n always denotes a positive integer.*

(3.10) (1)
$$
\begin{array}{c|c|c|}
 & Y_1 & Y_2 \\
\hline
X_1 & A^1 & 0 \\
\hline
x_1 & e & f \\
\hline
X_2 & D & A^2 \\
\hline
\end{array}
$$

(2)
$$
\begin{array}{c|c|c|c|}
 & Y_1 & y_1 & Y_2 \\
\hline
X_1 & A^1 & g & 0 \\
\hline
X_2 & D & h & A^2 \\
\hline
\end{array}
$$



The following statements hold for (1)–(6):

(3.11)

(a) *D does not span e or h, and f and g are nonzero.*

(b) *Let D^i be the submatrix of* (3)–(6) *defined by the column index of any α_i and all columns to the left of it, and the row index of α_i and all rows below it. Then* $\text{rank}(D^i) = \text{rank}(D) + 1$, *but for any proper submatrix $\bar{D}$ of D^i that does not contain α_i,* $\text{rank}(\bar{D}) \le \text{rank}(D)$. *In particular, $s^i = 0$ if $r^i = 0$, and $d^i = 0$ if $c^i = 0$.*

For completeness we insert here a polynomial algorithm that carries out the previously described shifting of rows and columns, and that finds one of the matrices (1)–(6) of (3.10) if the k-separation of N does not induce one for M.

Partitioning Algorithm

Input. B_M of (3.5) but without the sets $X_{31}, X_{32}, Y_{31}, Y_{32}$.

Output. Either: $X_{31}, X_{32}, Y_{31}, Y_{32}$ so that $(X_1 \cup Y_1 \cup X_{31} \cup Y_{31}, X_2 \cup Y_2 \cup X_{32} \cup Y_{32})$ is a k-separation of M induced by the k-separation $(X_1 \cup Y_1, X_2 \cup Y_2)$ of N.

Or: A submatrix of B_M of the form (1),(2),..., or (6) of Theorem 3.9.

Step 0. Initialize $X_{31} = Y_{31} = \emptyset$, $X_{32} = X_3$, $Y_{32} = Y_3$, and flag $= 0$. Throughout let $\tilde{D}, e(x), f(x), g(y), h(y)$ be the following submatrices of $B_M : \tilde{D}$ is defined by the rows of X_2 and the columns of the current $Y_1 \cup Y_{31}$; $e(x)$ $(f(x))$ is the row subvector specified by row x and the columns of the current $Y_1 \cup Y_{31}$ (the columns of Y_2); finally $g(y)$ $(h(y))$ is the column subvector given by column y and the rows of the current $X_1 \cup X_{31}$ (the rows of X_2).

Step 1. (Row Shifting) For each $x \in X_{32}$ do: If $e(x)$ is not spanned by $\tilde{D}$, shift x from X_{32} to X_{31}, set flag $= 1$, and examine $f(x)$; if the latter vector is nonzero, go to 4.

Step 2. (Column Shifting) For each $y \in Y_{32}$ do: If $g(y)$ is nonzero, shift y from Y_{32} to Y_{31}, set flag $= 1$, and examine $h(y)$; if the latter vector is not spanned by $\tilde{D}$, go to 4.

Step 3. (Termination Test) If $X_{32} \cup Y_{32} = \emptyset$ or flag $= 0$, stop; the desired sets $X_{31}, X_{32}, Y_{31}, Y_{32}$ have been found. Otherwise set flag $= 0$ and go to 1.

Step 4. (Partitioning Impossible) Delete all rows of X_{32} and columns of Y_{32} from B_M, then list the rows of X_{31} (the columns of Y_{31}) in the order in which they were shifted, i.e., the topmost row (leftmost column) of X_{31} (Y_{31}) was shifted first. A simple scanning scheme which processes the rows of X_{31} bottom to top and the columns of Y_{31} right to left, then extracts one matrix of type (1),(2),..., or (6) of (3.10) from B_M, and this matrix satisfies (3.11).

Suppose B_M itself is one of the matrices (1)–(6). Then B_M is minimal in the following sense: B_M cannot be partitioned, but deletion of any row of X_3 or any column of Y_3 makes a partition possible (for a proof just apply the partitioning algorithm). Note, however, that the minor N may be producible from M by a number of different reduction sequences, and that one may be able to find a smaller instance of (1)–(6) by searching through all the related partial representations. In general such a search seems to be difficult, but [13] contains a polynomial (but not practically usable for $k \geq 3$) test for the special case when (i) $\mathcal{M}$ is a subset of the class of matroids representable over a given finite field $\mathcal{F}$, (ii) each $M \in \mathcal{M}$ is specified by a standard representation matrix over $\mathcal{F}$, (iii) if N has two standard representation matrices over $\mathcal{F}$ with same support, then one matrix is a scaled version of the other one, and (iv) k is bounded by some given constant. If the given field is GF(2) or GF(3), the requirements of (ii) and (iii) are trivial. Thus an efficient test is always possible if $\mathcal{M}$ is contained in the class of binary or ternary matroids and k is bounded by some constant. Since we will not make use of this test, we omit details, which are included in [13]. There are, however, easily recognized instances where B_M cannot possibly be minimal. The simplest case is covered by the next

Lemma 3.12 below. That lemma and the next one rely on the following convention. Once a matrix B_M of Theorem 3.9 has been introduced, then it is implicitly assumed that M is the matroid that has $\hat{B}_M$ as partial representation, that N is the minor of M corresponding to the submatrix of B_M composed of $A^1, D, 0$, and A^2, and that the k-separation of N is $(X_1 \cup Y_1, X_2 \cup Y_2)$.

Lemma 3.12 [13]. *Suppose for a given B_M of Theorem 3.9 the k-separation of N induces a k-separation in every proper minor of M that in turn has N as a minor. Then in B_M*

(3.13)
- (a) *each subvector $c^i \neq 0$ unless it resides in the same column as subvector g;*
- (b) *each subvector $r^i \neq 0$ unless it resides in the same row as subvector f.*

For our decomposition algorithm (to follow shortly) we need more than Theorem 3.9 and Lemma 3.12. Specifically, let $M, N \in \mathcal{M}$ be as in Lemma 3.12. We then would like to have an efficient characterization of the minimal minors $\bar{M}$ of M that have a minor $N_1 \cong_{(L,F)} N$ such that *some* k-separation of N_1 which corresponds to the given k-separation of N under one of the (L, F)-isomorphisms, does not induce a k-separation of $\bar{M}$. Note that any such N_1 must be in $\mathcal{M}$ since $\mathcal{M}$ contains all matroids (L, F)-isomorphic to N. We have not been able to find such a characterization, but instead have identified easily checked and apparently useful necessary conditions that any minimal $\bar{M}$ must satisfy. These conditions are summarized in Lemma 3.14 below. They are almost identical to the conditions of Lemma 10.11 of [13], and it is a simple matter to adapt the proof of the latter lemma to the case at hand.

Lemma 3.14. *Let B_M be a matrix of type* (3), (4), (5), *or* (6) *of Theorem* 3.9, *and suppose the following holds for every proper minor $\bar{M}$ of M that in turn has a minor $N_1 \cong_{(L,F)} N$: Every k-separation of N_1 corresponding to (T_1, T_2) of N under one of the (L, F)-isomorphisms between N_1 and N, induces a k-separation of $\bar{M}$. Then any B_M of M of type* (3), (4), (5), *or* (6) *of* (3.10) *satisfies the following.*

(3.15)
- (a) *If e or r^i is not in the same row as f, then that vector is not parallel to any row $x \notin L$ of A^1 and is not a unit vector with* 1 *in a column $y \notin L$, provided y is not a loop of N/T_2.*
- (b) *Vector $[\frac{g}{c^1}]$ is not parallel to a column $y \notin L$ of $[\frac{A^1}{D}]$ if y is not a coloop of $N \setminus T_2$, and it is not a unit vector with* 1 *in a row $x \notin L$ if x is not a coloop in $N \setminus T_2$.*
- (c) *If h or c^j is not in the same column as g, then that vector is not parallel to a column $y \notin L$ of A^2 and is not a unit vector with* 1 *in a row $x \notin L$, provided x is not a coloop of $N \setminus T_1$.*
- (d) *Vector $[r^n | f]$, $n \geq 1$, is not parallel to a row $x \notin L$ of $[D | A^2]$ if x is*

not a loop of N/T_1, and it is not a unit vector with 1 *in a column $y \notin L$ if y is not a loop of N/T_1.*

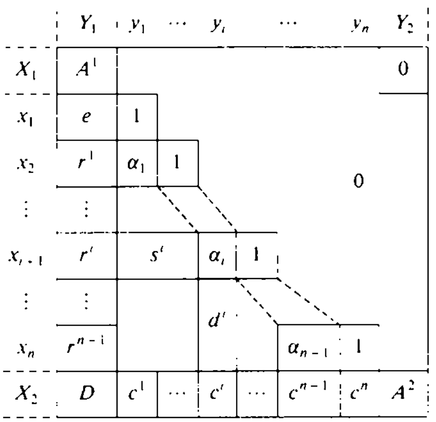
The next lemma relates induced k-separations to k-sums.

Lemma 3.16 [13]. *Suppose a matroid N is a proper k-sum, $k \geq 2$, say with partial representation $\hat{B}$, where B is the matrix of* (3.1). *If the k-separation $(X^1 \cup Y^1, X^2 \cup Y^2)$ of N induces a k-separation of a connected matroid M that has N as a minor, then M is a proper k-sum, which has* 3*-connected components if $k \geq 3$ and M is* 3*-connected.*

We now turn to the decomposition algorithm, which is a variant of the algorithm of [13]. The algorithm is based on the following ideas. Suppose we have a class $\mathscr{M}$ of matroids that is closed under the taking of minors, and a nonempty subclass $\mathscr{W}$, where each matroid in $\mathscr{W}$ is 3-connected and has at least four elements. Further suppose that a matroid of $\mathscr{W}$, say N, has a k-separation (T_1, T_2) as in (3.4) for some $k \geq 3$. In principle we could identify all minimal matroids $M \in \mathscr{M}$ such that (i) M has N as a minor, and (ii) there exists a minor $N_1 \equiv_{(L,F)} N$ of M such that a k-separation of N_1 that corresponds to (T_1, T_2) of N, does not induce a k-separation of M. Suppose we now replace N in $\mathscr{W}$ by these minimal matroids, getting, say, $\mathscr{V}$. A moment's reflection convinces us that the following claim is valid: For every $M \in \mathscr{M}$ at least one of the statements below holds: (1) M has no minor in $\mathscr{W}$; (2) M has a minor in $\mathscr{V}$; (3) M has a minor equal to N; for every such minor, say N_1, every k-separation of N_1 corresponding to (T_1, T_2) of N under one of the (L, F)-isomorphisms between N and N_1, induces a k-separation of M. We could repeat the above process, this time starting with $\mathscr{V}$ instead of $\mathscr{W}$, and thus could generate another potentially interesting claim. Indeed, one could continue until $\mathscr{V}$ becomes empty (a very attractive situation), or until one becomes tired of carrying out the computations. We omit details of the rather obvious inductive arguments and instead examine the first iteration (involving N) more closely. A generally formidable task in that iteration is the identification of the minimal matroids to be added to $\mathscr{W}$ upon removal of N. But we have a list of necessary conditions (3.10), (3.11), (3.13), (3.15), that such minimal matroids must observe, so we could add to $\mathscr{W}$ all matroids of $\mathscr{M}$ observing these conditions for N and its k-separation (T_1, T_2). Denote by $\tilde{\mathscr{V}}$ the collection of these matroids. Clearly $\tilde{\mathscr{V}}$ includes the minimal matroids, and the initial claim remains valid when we use $\mathscr{V} = (\mathscr{W} - \{N\}) \cup \tilde{\mathscr{V}}$. With this change we still face a second problem. The number of minimal matroids, and hence the number of matroids in $\tilde{\mathscr{V}}$, could be infinite or so large that just listing of the matroids cannot be carried out on any computer in reasonable time. A simple observation helps overcome this obstacle: the initial claim remains valid if we derive $\mathscr{V}$ from $\mathscr{W}$ by adding to $\mathscr{W} - \{N\}$ some *minor* of V, for every $V \in \tilde{\mathscr{V}}$. There are several ways to select such minors. Particularly attractive seem to be ones defined by the following matrices, which constitute certain submatrices of (3)-(6) of (3.10).

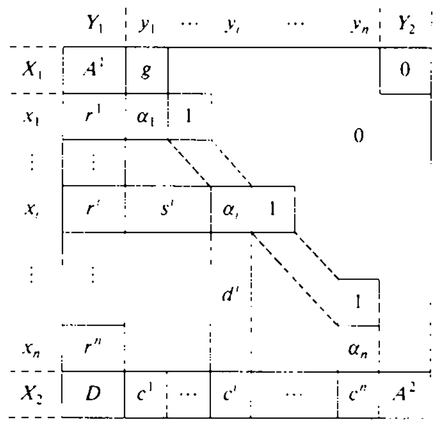
(3.17)

(7)



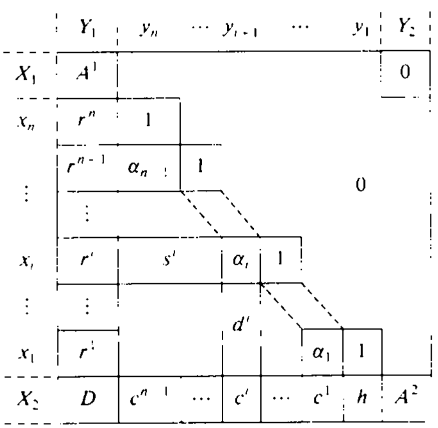
Column y_n may be absent

(8)



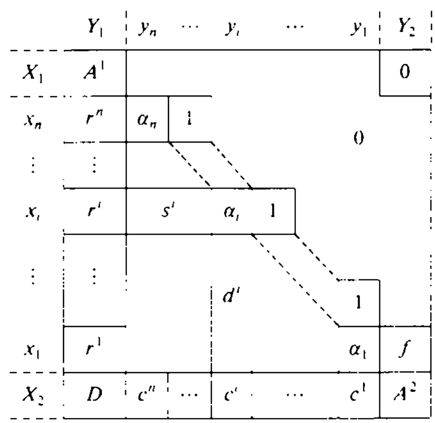
Row x_n may be absent

(9)



Row x_n may be absent

(10)



Column y_n may be absent

The derivation is as follows. Each matrix of (3)–(6) of (3.10) has a staircase structure indexed by the x_i and y_i. Each staircase starts with a row containing e, or a column containing g, and stops with a row containing f, or a column containing h. We then delete the last part of such a staircase to produce (7) or (8) of (3.17), or delete the first part to obtain (9) or (10). Note that we use a slightly different indexing of the partial staircase in case of (9) and (10) to simplify the subsequent discussion. The matrices of (3.17) so derived from (3.10) obviously satisfy (3.11), (3.13), and (3.15) as well. From now on we assume that all matrices of (3.10) and (3.17) specified below *do satisfy these conditions without explicitly saying so.* To strengthen the effectiveness of the conditions of (3.15), we will only consider k-separations as in (3.4) where (3.18) or (3.19) below hold.

(3.18) A^1 has no zero column (equivalently: N/T_2 has no loops).

(3.19) A^2 has no zero rows (equivalently: $N \setminus T_1$ has no coloops).

One is tempted to call a matroid defined via (3.10) a *complete violator* of the property of induced k-separation, and one defined via (3.17) a *partial violator*. For brevity we often drop the word 'violator', and thus talk about *complete* and *partial* matroids. With these terms we can rephrase the previously described process as follows: Replace N of $\mathcal{N}$ by a number of matroids V, where each V is complete or partial, to get the set $\mathcal{V}$.

It is possible that no matroid of the initial set $\mathcal{N}$ has a k-separation of type (3.4), for any k, that we would want to use. If we still want to proceed in some way, we can derive a $\mathcal{V}$ from $\mathcal{N}$ by replacing an N of $\mathcal{N}$ by certain 3-connected 1-, 2-, or 3-element extensions V of N. The V's are so chosen that we can apply Theorem 2.2 or 2.3 and claim that any $M \in \mathcal{M}$ with N as a proper minor has one of the V's as a minor, or is 2-separable.

The situation becomes more complicated when we attempt to apply the above construction recursively, since then we must consider the situation where the current collection of matroids contains complete and partial violators. Before proceeding to this more elaborate case, we introduce additional terminology to simplify the exposition.

Let $\bar{S}$ and $\bar{T}$ be subsets of same cardinality of the groundsets of N_1 and N_2 in $\mathcal{M}$, respectively. Then $\bar{S}$ *corresponds to* $\bar{T}$ if there exists an (L,F)-isomorphism from N_1 to N_2 that maps $\bar{S}$ onto $\bar{T}$. Statements like "the k-separation (S_1, S_2) of N_1 corresponds to the k-separation (T_1, T_2) of N_2" are analogously interpreted. We also will no longer be interested in any difference between two (L,F)-isomorphic matroids for the remainder of this section, and hence will consider them to be equal. Note that this notion of 'equal' depends on L *and* F.

The recursive construction involves successive enlargement of a directed, acyclic *decomposition graph*, where each node M corresponds to a complete or partial matroid M. Correspondingly we call each node *complete* or *partial*. The initial graph, which represents the set $\mathcal{N}$, contains no arcs, and all nodes are declared to be complete as a matter of convenience.

In each iteration we process a node not examined so far. Such a node is called *open*. We then create new open nodes, add arcs without introducing a directed cycle, and finally declare the currently processed node to be *closed*. The algorithm stops when all nodes have become closed, a very attractive situation, or when we tire of the computations. While the algorithm proceeds, we have numerous choices to make, the effects of many of which are not quite clear at the time they come up. Thus the algorithm is by no means a purely deterministic process, but generally requires intuitive insight into the structure of the matroid class $\mathcal{M}$ at hand. As a demonstration of this fact one need only compare the decomposition theorem 4.3 below with the complicated theorem of [7,9]. The difference is solely due to a different choice in the third iteration of the algorithm. Also note that the decomposition algorithm makes no use of the partitioning algorithm described earlier in this section. We will use the scheme, though, to establish that certain testing can be done in polynomial time, in the proof of Corollary 3.22.

The decomposition algorithm is a bit easier to describe and understand when one introduces a subroutine that handles most of the processing of an open node. We give this subroutine first.

Subroutine

Input. $\mathcal{M}$, a class of matroids whose groundsets always contain a given (possibly empty) set L. A set F of functions, $F \subseteq \{f \mid f$ is a bijection from L to $L\}$, that includes the identity function if $L \neq \emptyset$, and that is closed under the taking of inverse. If $M \in \mathcal{M}$, then all minors of M containing L are also in $\mathcal{M}$. An open node N of a decomposition graph, where $N \in \mathcal{M}$. A matrix B of (3.17) for N if node N is partial.

Output. A list of complete violators. A second list of partial violators, together with the related matrices of (3.17). Either list may be empty.

Procedure.

Step 0. If node N is partial, go to step 2.

Step 1. Attempt to find a matrix $B = B_N$ of (3.4) for matroid N that satisfies $X_i, Y_i \neq \emptyset$, $i = 1, 2$, (3.18) or (3.19), plus possibly some additional conditions, e.g., concerning position of the elements of L relative to the sets X_i and Y_i, $i = 1, 2$. If it is computationally unattractive, infeasible, or impossible to locate such a matrix, go to Step 3.

Step 2. Determine all matrices of (3.10) and (3.17) that may be obtained from B by adding exactly one row or one column. If B was produced by Step 1, then of the possible matrices of (3.17) one need only produce either the ones of type (7), (8), or those of (9), (10). The choice may be made according to any criterion, except that (3.18) must hold in case of (7), (8), and (3.19) in case of (9), (10). Let $N_1, N_2, \ldots$ be the related complete violators produced via (3.10), and $(M_1, B^1), (M_2, B^2), \ldots$ be the pairs of partial violators and matrices generated via (3.17). Go to Step 4.

Step 3. Let $\mathcal{V}_j = \{V \mid V$ is a 3-connected j-element extension of $N\}$, $j = 1, 2, 3$.

(a) If $|L| \leq 1$, let $N_1, N_2, \ldots$ be the matroids of $\mathcal{V}_1$. If N is a wheel (whirl), also include in the list of the N_i any next larger wheel (whirl) with N as a minor.

(b) If $|L| > 1$, let $N_1, N_2, \ldots$ be the matroids of $\bigcup_{j=1}^{3} \mathcal{V}_j$.

Step 4. Delete from the lists any N_i or (M_i, B^i) where the matroid is not in $\mathcal{M}$. Delete additional N_i to eliminate (L,F)-isomorphic instances except for one representative of each (L,F)-isomorphism class. Similarly delete (M_i, B^i) to eliminate instances of special (L,F)-isomorphisms, each of which must satisfy the following condition. The bijection establishing the (L,F)-isomorphism must map $X_i \cup Y_i$ of one matroid onto $X_i \cup Y_i$ of the other one, for $i = 1, 2$, and must be an identity on the remaining x_j and y_j elements. The deletions described above need only be carried out as far as it is computationally attractive or possible to identify them. However, enough deletions must be made so that the resulting two lists are finite. Upon renumbering we may presume the two lists to be $N_1, N_2, \ldots, N_r$ and $(M_1, B^1), (M_2, B^2), \ldots, (M_s, B^s)$, for some r and s. These lists constitute the output.

Steps 1 and 3 and most material of Step 2 are obviously motivated by the informal discussion at the beginning of this section, so we will only add some explanations about Step 4 and part of Step 2. In the latter step far fewer choices exist for generating matrices of (3.10) and (3.17) than might appear from the description. For example, if B is a matrix of (3.4), then for (3.10) only (1) and (2) are possible, and for (3.17), either (7) with just x_1 and (8) with just y_1, or (9) with just y_1 and (10) with just x_1. As a second example, let B be a matrix of (7) of (3.17) with x_1 and y_1. Then (5) with x_1, y_1, and x_2, is the only possible case of (3.10), and we may only have (7) with x_1, y_1, and x_2 for (3.17). Introduction of the restrictive definition of (L, F)-isomorphism for the partial cases of (3.17) may appear to be unnecessary, but actually is essential for the proof of the main decomposition theorem. Testing for such a special (L, F)-isomorphism, say of M_1 and M_2, is equivalent to the following: Check whether or not the partial representation $\hat{B}^1$ of M_1 is obtainable from $\hat{B}^2$ for M_2 by pivots within A^1 and/or A^2, and by possibly repeated use of the following operation: exchange two columns or two rows both of whose indices are in $X_i \cup Y_i$, for $i = 1$ or 2.

We are now ready for the decomposition algorithm.

Decomposition Algorithm

Input. $\mathcal{M}$, a class of matroids whose groundsets always contain a given (possibly empty) set L. A set F of functions, $F \subseteq \{f \mid f$ is a bijection from L to $L\}$, that includes the identity function if $L \neq \emptyset$, and that is closed under the taking of inverse. If $M \in \mathcal{M}$, then all minors of M containing L are also in $\mathcal{M}$. A subset $\mathcal{W}$ of $\mathcal{M}$, where each matroid is 3-connected and has four or more elements.

Procedure

Step 0. (Initialization) Define $\mathcal{N} = \mathcal{S} = \emptyset$. For each matroid in $\mathcal{W}$ create a node, which is declared to be open and complete. These nodes, without any connecting arcs, constitute the initial decomposition graph $\mathcal{X}$.

Step 1. (Select another open node) If all nodes of $\mathcal{X}$ are closed, stop. Otherwise select an open node N. If node N is partial, the node also specifies a matrix B of (3.17).

Step 2. (Process open node N) Execute the subroutine with $\mathcal{M}$, L, F, open node N, and B if applicable, as input, to get two lists, $N_1, N_2, \ldots, N_r$, and $(M_1, B^1), (M_2, B^2), \ldots, (M_s, B^s)$. If N is complete: add N to $\mathcal{N}$ if a k-separation of type (3.4) was found for N in Step 1 of the subroutine, and add N to $\mathcal{S}$ otherwise. N is not added to either set if N is partial.

Step 3. (Update decomposition graph $\mathcal{X}$) let $\mathcal{R}$ be the set of complete (open or closed) nodes of $\mathcal{X}$ from which there is no directed path to node N. Process each member N_i or (M_i, B^i) of the two lists as follows, as far as is computational feasible or attractive: If a member R of $\mathcal{R}$ is a minor of N_i *or* M_i, then delete N_i or (M_i, B^i) from the list, and add to $\mathcal{X}$ a directed arc from node N to node R. Once as many reductions as possible or desired have been made, create a new open node for each remaining entry in the two lists. Such a new node is complete and labelled N_i for any N_i, and is partial and labelled M_i for any (M_i, B^i). In the latter case we also record B^i with node M_i. Finally a directed arc is added from node N to each of the nodes just created, and node N is declared closed. Define $\mathcal{V}$ to be the set of open nodes of $\mathcal{X}$, and go to Step 1.

The following conclusions may be drawn at the end of each iteration through Steps 1–3 of the algorithm.

Theorem 3.20. *Suppose one has performed any number of iterations through Steps* 1–3 *of the decomposition algorithm, and that one just has completed Step* 3. *Also assume that the decomposition graph does not have an infinite subset of nodes such that the cardinality of the groundsets of these nodes is uniformly bounded by some constant. Then the sets* $\mathcal{N}$, $\mathcal{S}$, $\mathcal{V}$, *and* $\mathcal{W}$ *in existence at that time, together with the matrices* B_N, $N \in \mathcal{N}$, *found in Step* 1 *of the subroutine, may be utilized to produce the following theorem and corollary.*

Theorem 3.21. *Every* 3*-connected* $M \in \mathcal{M}$ *with four or more elements obeys one of the conditions below.*

(1) *M has no minor in* $\mathcal{W}$.

(2) *M has a minor in* $\mathcal{V}$.

(3) *M is equal to some* $N \in \mathcal{S}$.

(4) *M has a minor in* $\mathcal{N}$, *say N, for which the following holds.* (i) *Every k-separation of every minor* $N_1 \cong_{(L,F)} N$ *induces a k-separation of M as long as the k-separation of* N_1 *corresponds to the k-separation* (T_1, T_2) *of N defined via* B_N. (ii) *Each such induced k-separation of M can be turned into a proper k-sum decomposition with* 3*-connected components provided* B_N *can be further subdivided to become a matrix of* (3.1) *that shows N to be a proper k-sum.*

Corollary 3.22. *Assume that a polynomial algorithm exists that either determines for a given* $M \in \mathcal{M}$ *that M has no minor in* $\mathcal{W}$, *or produces a minor of M in* $\mathcal{W}$. *Further assume that only a finite number of iterations have been done with the decomposition algorithm, and that all lists produced by the subroutine are available as well as the current decomposition graph. Then there is a polynomial algorithm that for any* 3*-connected* $M \in \mathcal{M}$ *on at least four elements either determines an applicable case* (1)–(3) *of Theorem* 3.21, *or locates a minor* $N \in \mathcal{N}$ *such that the k-separation* (T_1, T_2) *of N as implied by* B_N *(which is of type* (3.4)*) induces a k-separation of M. Such an induced k-separation can be demanded to be a proper k-sum decomposition with* 3*-connected components provided the matrix* B_N *can be further subdivided to become a matrix of* (3.1) *that shows N to be a proper k-sum.*

We omit the proof of Theorem 3.20 since it is almost identical to the one of Theorem 11.5 of [13] once the following lemma has been established.

Lemma 3.23. *Let M be a matroid with a partial representation $\hat{B}$ whose B is one of the matrices* (1)–(6) *of* (3.10). *In each case the submatrix B_N composed of $A^1, 0, D$, and A^2 (each of which is assumed nonempty) is supposed to correspond to a* 3*-connected matroid N. Furthermore* (3.11) *and* (3.13) *are to hold as well as* (3.18) *or* (3.19). *Then M is* 3*-connected. In particular the matroids of complete nodes of $\mathscr{X}$ and of $\mathscr{N}$ and $\mathscr{S}$ contain only* 3*-connected matroids.*

Proof. Let m be the number of elements M has beyond those of N. Simple checking proves the lemma for $m = 1$ or 2, so by induction we may assume M to be 3-connected whenever m does not exceed some $m' \geq 2$. Take now an instance with $m' + 1$ elements. By duality we may suppose that (3.18) holds, so A^1 has no zero column.

If B is of type (3) *or* (5): If subvector e is a unit vector, perform a pivot in A^1 to change it to a non-unit vector. If e is parallel to a row of A^1, say with index $x \in X_1$, then remove row x from B, and put row x_1 in its place. We thus get a smaller instance (e.g., (5) becomes (4)), and the related matroid is 3-connected by induction. Now adjoin row x again, and M is seen to be 3-connected as well. Finally if e is not a unit vector and is not parallel to a row of A^1, then expand N by x_1, or equivalently, adjoin $[e|0]$ to B_N. The new minor of M is also 3-connected, and by induction M must be 3-connected as well.

If B is of type (4) *or* (6): By (3.13), r^1 is nonzero. If r^1 is a unit vector, we change it to a non-unit vector by a pivot in A^1. If r^1 is parallel to a row x of A^1, we delete row x from B, and put row x_1 in its place. By induction $M \setminus y_1$ is then 3-connected, and so must be M. Finally assume r^1 is not a unit vector and is not parallel to a row of A^1. By arguments analogous to those for e above, $M \setminus y_1$ and hence M, must then be 3-connected. $\square$

The proof of Corollary 3.22 is essentially the same as that of Corollary 11.8 of [13], but we do include it here since it outlines the claimed polynomial algorithm. The proof proceeds as follows.

Given a 3-connected $M \in \mathscr{M}$ on at least four elements, we use the assumed polynomial algorithm to determine that M has no minor in $\mathscr{W}$, or to find a minor, say N, in $\mathscr{W}$. In the former case we are done, while in the latter we trace through the current decomposition graph $\mathscr{X}$, which is easily proved to be acyclic, as follows. We start at node N, which is complete. If N was never selected in Step 1, then N is open, and hence in $\mathscr{Y}$, and we are done. Otherwise we examine the iteration of the algorithm when N was processed. If in the subroutine a k-separation was determined for N, then we first check with the partitioning algorithm described earlier, in polynomial time whether or not that k-separation induces one for M. In the affirmative case we are done, since the k-sum requirement is easily met using the shifting algorithm of Theorem 3.3 of [11]. Otherwise the partitioning algorithm produces a complete violator V. Now one of the N_i or M_i (of an (M_i, B^i)) in the lists produced by the subroutine must be (L, F)-isomorphic to a minor of V. In constant time we locate that minor, and also the related B^i in case of an M_i. In the latter case M_i is related to the minor via one of the special (L, F)-isomorphisms defined in the subroutine. Now this N_i or M_i is a descendent node of N, or there is an arc from N to a node whose matroid is a minor of N_i or M_i. In constant time we settle which case applies, and then proceed inductively with almost identical arguments. Thus we only need to cover the third case, where $N \in \mathscr{S}$. If $M = N$, we are done. Otherwise the subroutine produced $N_1, N_2, \ldots, N_r$, one of which must be (L, F)-isomorphic to a 3-connected 1-, 2-, or 3-element extension of M. The latter minor of M can be found in polynomial time using [10], and selection of the correct N_i is done in constant time. The rest is again handled by induction. The number of times any of the above-described computational steps is performed is bounded by the number of arcs of the decomposition graph. But the latter number is some constant since the decomposition algorithm has run for a finite number of iterations. Thus total effort until the above-described scheme stops, is bounded by a polynomial in the size of M. We should note that the set $\mathscr{W}$ is allowed to be infinite. In that case we do not store any open nodes N, $N \in \mathscr{W}$, and stop with the conclusion "M has a minor in $\mathscr{W}$" when the initial polynomial algorithm (which tests whether or not M has a minor in $\mathscr{W}$) produces a matroid of $\mathscr{W}$ that is not explicitly listed.

We should mention that applications of the algorithm with infinite number of iterations *are* of interest, and that the condition of infinite node subsets in Theorem 3.20 prevents pathological constructions where an infinite number of isomorphic matroids are generated. In Corollary 3.22 we have confined outselves to the case of finite number of iterations since the conditions we have developed for the infinite case are rather technical. However, these conditions can sometimes be significantly simplified for special matroid classes as we will show in a subsequent paper.

4. Application of the decomposition algorithm

In this section we apply the decomposition algorithm to the class $\mathscr{M}$ of binary matroids mentioned in the Introduction, i.e., $\mathscr{M} = \{M \mid M$ is binary, has an element l, and l is not in any F_7^* minor of $M\}$. From now on all matrices are over GF(2), and terms like determinant, pivot, rank, etc., are to be interpreted correspondingly.

Before going into details of the iterations of the algorithm it seems worthwhile that we motivate some choices. Goal is to characterize $\mathscr{M}$ by a decomposition theorem of the form of Theorem 3.21. Any regular matroid with an element l occurs in $\mathscr{M}$, by Tutte's characterization [15], and its structure is well understood due to Seymour's work [6]; see also [13]. Thus one should concentrate on nonregular matroids in $\mathscr{M}$. If such a matroid is 2-separable, the analysis is easy, as is shown in the next section. Thus we are led to focus on 3-connected nonregular matroids in $\mathscr{M}$. A natural choice for $\mathscr{W}$ of the decomposition algorithm are then any minimal matroids of that type. Surprisingly there is only one such matroid, up to l-isomorphism, as shown in the following lemma. Since differences between l-

isomorphic matroids of $\mathcal{M}$ are of no interest to us here, we will consider such matroids to be equal for the remainder of this paper. Clearly all matroids of $\mathcal{M}$ that are isomorphic to F_7 are l-isomorphic, so we may use 'F_7' to designate any such matroid without any risk of confusion.

Lemma 4.1. *For any 3-connected nonregular matroid $M \in \mathcal{M}$ the following holds.*

(a) *M has a minor equal to $F_7 \in \mathcal{M}$.*

(b) *If M has more than seven elements, it has a minor equal to the matroid N_1 represented by $\hat{E}$ with*

(4.2) $E=$

	x	y	z	l
d	1	1	1	0
a	1	0	1	1
b	1	1	0	1
c	0	1	1	0

The indicated partition of E corresponds to a 3-sum decomposition of N_1.

Proof. By [15], M has an F_7 or F_7^* minor. If only one of these occurs, M must be equal to F_7 with an element l by the splitter result of [6] and the fact that $M \in \mathcal{M}$. Hence we may suppose that M has eight or more elements and an F_7^* minor. By Lemma 2.13 and the fact that $M \in \mathcal{M}$, M contains a 3-connected 1-element extension of F_7^* such that $l \notin F_7^*$. Simple checking reveals that N_1 is the only such matroid. Finally the partition of E obviously corresponds to a 3-sum decomposition, and N_1 has $F_7 \in \mathcal{M}$ as a minor. ❑

We now apply the decomposition algorithm. $\mathcal{M}$ is the class just defined, $L = \{l\}$, $F = \{\text{identity function}\}$, and $\mathcal{N} = \{F_7\}$. In Step 0 we define $\mathcal{S} = \mathcal{T} = \emptyset$, and let $\mathcal{X}$ be a graph with just one node, which is open and labelled F_7.

Iteration 1

Step 1. Open node F_7 is the only choice.

Step 2. F_7 has no k-sum decomposition for any k, and we opt for Step 3 of the subroutine. By Lemma 4.1 the lists of the N_i produced in Steps 3 and 4 of the subroutine consists just of N_1 defined by (4.2). The second list of type (M_i, B^i) is empty. We add F_7 to $\mathcal{T}$.

Step 3. We add to $\mathcal{X}$ an open complete node labelled N_1 as well as a directed arc from F_7 to N_1, and declare node F_7 to be closed.

Iteration 2

Step 1. Open and complete node N_1 is the only possible choice.

Step 2. In Step 1 of the subroutine we select a 3-separation of N_1 as given by (4.2), i.e.

$$B_{N_1} = \begin{array}{cc|ccc|c|} & & & Y_1 & & Y_2 \\ & & x & y & z & l \\ \hline X_1 & d & 1 & 1 & 1 & 0 \\ \hline & a & 1 & 0 & 1 & 1 \\ X_2 & b & 1 & 1 & 0 & 1 \\ & c & 0 & 1 & 1 & 0 \\ \hline \end{array}$$

This matrix satisfies (3.18), but not (3.19).

Candidates for (1) of (3.10) have a row $[e|f]$ where e is not spanned by D, and f is nonzero. Thus e is a unit vector or has three 1's, and $f=[1]$. In each case l can be placed into an F_7^* minor.

Candidates for (2) of (3.10) have a column $[\frac{g}{h}]$ such that g is nonzero and h is not spanned by D. Thus $g=[1]$, and h is a unit vector or has three 1's. The four cases are l-isomorphic, so anticipating Step 4 of the subroutine we only retain N_2 with

$$B_{N_2} = \begin{array}{c|ccc|c|c|} & x & y & z & e & l \\ \hline d & 1 & 1 & 1 & 1 & 0 \\ \hline a & 1 & 0 & 1 & 1 & 1 \\ b & 1 & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 1 & 0 & 0 \\ \hline \end{array}$$

as the only matroid for the list of complete violators. Note that each of the l-isomorphisms maps e onto e, so each of them becomes an l-automorphism of N_1 when restricted to N_1. The list of partial violators turns out to empty due to the following arguments. Since B_{N_1} satisfies (3.18), but not (3.19), we must consider all B_V of type (7) and (8) of (3.17) having just one more row or column than B_{N_1}.

(7): The added row is $[e|0]$, where e is not spanned by D by (3.11). Thus e is a unit vector or has three 1's. But each of these cases can be eliminated by (3.15)(a).

(8): The added column is $[\frac{g}{c^1}]$, where g is nonzero and c^1 is spanned by D by (3.11). But each of the possible cases is ruled out by (3.15)(b). Just before exiting Step 2 we add N_1 to $\mathcal{A}$.

Step 3. We add to $\mathcal{X}$ an open complete nodel labelled N_2 as well as a directed arc from N_1 to N_2, and declare node N_1 to be closed.

Iteration 3

Step 1. Open and complete node N_2 is the only choice.

Step 2. In Step 1 of the subroutine we select a 3-separation of N_2 as given by

$$B_{N_2} = \begin{array}{c|ccc|cc|} & a & d & x & l & e \\ \hline b & 1 & 1 & 1 & 0 & 0 \\ \hline z & 1 & 0 & 1 & 1 & 1 \\ y & 1 & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 1 & 0 & 1 \\ \hline \end{array}$$

Note that B_{N_2} implies a 3-separation $(\{b,a,d,x\},\{z,y,c,l\})$ of $N_2 \setminus e = N_1$. This 3-separation is different from the one of B_{N_1}, i.e., from $(\{d,x,y,z\},\{a,b,c,l\})$. However, an l-automorphism of N_1 maps $\{b,a,d,x\}$ and $\{z,y,c,l\}$ onto $\{d,x,y,z\}$ and $\{a,b,c,l\}$, so we may obtain the 3-separation of N_2 from the one of N_1 in Iteration 2 using that l-automorphism. Matrix B_{N_2} satisfies both (3.18) and (3.19). We now show that two empty lists are produced in Steps 2 and 3 of the subroutine. Candidates for (1) of (3.10) must have a row $[e\,|\,f]$ where e is not spanned by D, and f is nonzero. Thus e is a unit vector or has three 1's, and $f=[10]$, $[01]$, or $[11]$. In each case l can be placed into an F_7^* minor. Candidates for (2) of (3.10) have a column $[\frac{g}{h}]$ such that g is nonzero and h is not spanned by D. Thus $g=[1]$ and h is a unit vector or has three 1's. Again in each case l occurs in an F_7^* minor. We conclude that the list of complete violators is empty. For computation of the list of partial violators, we may choose (7) and (8), or (9) and (10), of (3.17) since both (3.18) and (3.19) are satisfied. We opt for (7)/(8), each with one additional row/column beyond those of B_{N_2}.

(7): The added row is $[e\,|\,0]$, where e is not spanned by D by (3.11). But (3.15)(a) rules out all such cases.

(8): The added column is $[\frac{g}{c^1}]$, where g is nonzero and c^1 is spanned by D. But (3.15)(b) eliminates all cases. Hence the list of partial violators is empty. Before leaving Step 2, we add N_2 to $\mathcal{A}$.

Step 3. Node N_2 is declared closed, and thus $\mathcal{Y}$ is empty.

The algorithm stops in Step 1 of the next iteration since all nodes are closed.

Theorem 3.21 plus the observation in Iteration 3 (concerning the relationship between N_1 and N_2) permit the following conclusion.

Theorem 4.3. *Every connected $M \in \mathcal{M}$ obeys one of the conditions below.*

(1) *M is regular.*

(2) *M is 2-separable.*

(3) *$M = F_7$.*

(4) *M is 3-connected and has a minor that is l-isomorphic to the matroid N_1*

defined via E of (4.2). *Let N be any such minor. Relabel the elements of M so that the elements of N are now labelled according to* (4.2). *Then at least one of the following three* 3*-separations of N induces a* 3*-separation of M.*

	T_1	T_2
1	d, x, y, z	a, b, c, l
2	a, b, d, x	c, y, z, l
3	a, b, y, z	c, d, x, l

Furthermore, any such induced 3*-separation of M can be converted to a proper* 3*-sum or to a semi-proper* 3*-sum with connected* A^1; *in either case* $l \in Y_2$. *Indeed, M has a representation* $\hat{B}$ *whose B is*

$$B = \begin{array}{c|ccc|c|c} & & Y_1 & & l & \longleftarrow Y_2 \longrightarrow \\ \hline X_1 & & A^1 & & & 0 \\ & 1 & 1 & 1 & 0 & \\ \hline X_2 & 1 & 0 & 1 & 1 & A^2 \\ & 1 & 1 & 0 & 1 & \\ & & D & & & \\ \hline \end{array} \tag{4.4}$$

where A^1 *is connected,* A^2 *is connected or equal to* $[1\ 1\ 0]^t$, *and* $\operatorname{rank}(D) = 2$.

Proof. Suppose that (2) does not hold. We then use Theorem 3.21 with the sets just produced by the decomposition algorithm. Statement (1) of that theorem plus Lemma 4.1 imply (1) here. Statement (2) of Theorem 3.21 is vacuous since $\mathcal{Y} = \emptyset$, while (3) of that theorem is (3) here. Thus (4) of Theorem 3.21 remains, and N_1 or N_2 must be a minor of M and must induce a 3-separation. Examination of the iterations of the algorithm reveals that N_2 can be eliminated provided three 3-separations of N_1 are considered. The arguments are as follows.

Let N be any minor of M l-isomorphic to N_1, and relabel the elements of M so that the elements of N are as given by E of (4.2). Suppose the 3-separation $(\{d, x, y, z\}, \{a, b, c, l\})$ of N does not induce a 3-separation of M. By Iteration 2, M then has a minor N_2 that is derived from N by adding an element, say e, to N. Four such additions are possible, as described in Step 2 of Iteration 2. The four matroids are l-isomorphic to the N_2 specified in Step 3 as follows:

h = first unit vector: this gives N_2.

h = second unit vector: the l-isomorphism takes $a, b, c, d, e, x, y, z, l$ of N_2 to $b, a, c, d, e, x, z, y, l$, respectively.

h = third unit vector: the l-isomorphism takes $a, b, c, d, e, x, y, z, l$ of N_2 to $a, b, c, z, e, y, x, d, l$, respectively.

h has three 1's: the l-isomorphism takes $a, b, c, d, e, x, y, z, l$ of N_2 to $a, b, c, y, e, z, d, x, l$ respectively. Note that in each case e of N_2 is mapped onto e of the second matroid, and that each l-isomorphism becomes an l-automorphism for N when restricted to the groundset of N.

In Iteration 3 it is shown that the 3-separation $(\{b, a, d, x\}, \{z, y, c, e, l\})$ of N_2 must induce a 3-separation of M. But then the 3-separation $(\{b, a, d, x\}, \{z, y, c, l\})$ of $N = N_2 \setminus e$ must also induce a 3-separation of M. This case corresponds to line 2 of the table in Theorem 4.3. Now instead of N_2 itself we may have produced one of the three l-isomorphic cases in Iteration 2. The related 3-separations of N are given by line 3, or they duplicate the cases of lines 1 and 2. By (4) of Theorem 3.21 the 3-separation of N given by line 1, or one of the 3-separations of N given by lines 1, 2, or 3 (which correspond to all ways in which N_2 could be derived from N in Iteration 2), must induce a 3-separation of M. This implies that M has a representation $\hat{B}$ whose B is given by (4.4), except possibly for the claims about A^1 and A^2. With the shifting algorithm of [11] we now produce a B satisfying the latter claims. If A^1 is not connected, it nevertheless must have a block (= maximal connected submatrix) of length 4 or more due to the submatrix [1 1 1]. Shift all other rows (columns) of X_1 (Y_1) to X_2 (Y_2). Thus without loss of generality we may suppose that A^1 is connected. Now A^2 must contain exactly one block since from two or more blocks we could produce a submatrix

$$\bar{A}^2 = \begin{array}{c} l \\ \begin{array}{|cc|} \hline 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ \hline \end{array} \end{array}$$

such that the submatrix of D specified by the first (last) two rows of $\bar{A}^2$, has rank equal to 2. It is then easily seen that l could be placed into an F_7^* minor, a contradiction. Thus we may shift rows (columns) of X_2 (Y_2) to X_1 (Y_1) until either A^2 has been reduced to a block of length at least 4, having l as index of one of the columns, or A^2 has beomce $[1\ 1\ 0]^t$, with l as column index. The new A^1 must still be connected, so both cases represent desired outcomes. □

While going over the iterations of the algorithm, the reader has surely noticed that in Iteration 3 more than one choice existed for continuing the scheme, and that by some selection process we apparently made a choice leading to a quick termination. This seemingly amazing insight on our part has a very simple explanation. Originally we did *not* know which choice to make, and as a result the algorithm ran for 21 iterations involving laborious manual calculations. The theorem so produced is included in [7,9]. Once the latter theorem was at hand, we looked for choices that would reduce the tremendous computing effort. The iterations described here constitute the outcome of this search.

In the next section, we apply Theorem 4.3 recursively to derive a 3-sum decomposition $M_1 \oplus_3 M_2$ of any 3-connected $M \in \mathcal{M}$ for which M_1 has certain properties.

5. A second decomposition theorem

Repeated application of Theorem 4.3 allows the following conclusion, where $\mathcal{M} = \{M \mid M$ is binary, has an element l, and l is not in any F_7^* minor of $M\}$ as before.

Theorem 5.1. *For any 3-connected $M \in \mathcal{M}$ at least one of the statements* (1)–(4) *below holds.*

(1) *M is regular.*

(2) *$M = F_7$.*

(3) *$M = M_1 \oplus_3 M_2$, where M_1 is a 3-connected regular matroid, or $M_1 = F_7$. In either case a B of type* (3.1) *displaying the 3-sum may be so chosen that $\bar{Y}_2 = \{l\}$.*

(4) *M has a triad T such that $l \notin T$.*

The following two lemmas will be used in the proof of the theorem.

Lemma 5.2. *Let $\hat{B}$ be a representation of a 3-connected binary matroid M where*

$$B = \begin{array}{c|c|c|}
 & Y_1 & Y_2 \\ \hline
X_1 & A^1 & 0 \\ \hline
X_2 & D & A^2 \\ \hline
\end{array}\;; \qquad \operatorname{rank}(D) = 2; \qquad \textit{all } X_i, Y_i \neq \emptyset. \tag{5.3}$$

Then $M = M_1 \oplus_3 M_2$ where M_1 and M_2 are represented by binary $\hat{B}^1$ and $\hat{B}^2$ with

$$B^1 = \begin{array}{c|c|c|}
 & Y_1 & g \\ \hline
X_1 & A^1 & 0 \\ \hline
e & a & 1 \\ \hline
f & b & 1 \\ \hline
\end{array} \quad \textit{and} \quad B^2 = \begin{array}{c|c|c|c|}
 & y & z & Y_2 \\ \hline
x & 1 & 1 & 0 \\ \hline
X_2 & c & d & A^2 \\ \hline
\end{array} \tag{5.4}$$

Vectors a and b (c and d) are two linearly independent rows (columns) of D, and $e, f, g \in X_2 \cup Y_2$ $(x, y, z \in X_1 \cup Y_1)$.

Proof. Without loss of generality we may assume that a maximal connected submatrix of A^1 (A^2) of B contains a submatix $[1\ 1]$ $([1\ 1]^t)$ such that the two columns (rows) of D below (to the left of) the latter matrix are linearly independent. If such matrices are not present, they may be produced by pivots within A^1 and/or A^2 of $\hat{B}$ (details of the pivot selection are given in the proof of Lemma 3.5 of [11]); any such pivots are also performed in $\hat{B}^1$ and/or $\hat{B}^2$. Now choose c and d (a and b) to be these linearly independent columns (rows) of D. They intersect on a 2×2 nonsingular submatrix $\bar{D}$ of D, so $M = M_1 \oplus_3 M_2$ when $e, f, g \in X_2 \cup Y_2$ and $x, y, z \in X_1 \cup Y_1$ are properly selected. □

Lemma 5.5. *Let M, M_1, M_2 be as in Lemma 5.2, and assume that every triad of M contains at least one element of $X_2 \cup Y_2$. If M_1 is $(3+)$-separable, then M is also a 3-sum $\tilde{M}_1 \oplus_3 \tilde{M}_2$ where $\tilde{M}_1$ is 3-connected and has fewer elements than M_1. Indeed, denote by $\tilde{B}$, $\tilde{B}^1$, and $\tilde{B}^2$ the matrices of (5.3) and (5.4) with tildes added to all symbols, and suppose these matrices correspond to the 3-sum $\tilde{M}_1 \oplus_3 \tilde{M}_2$. Then $\tilde{X}_1 \cup \tilde{Y}_1 \subset X_1 \cup Y_1$, and $\tilde{B}$ can be derived from B by pivots in A_1 and subsequent repartitioning.*

Proof. Denote by (S_1, S_2) the assumed $(3+)$-separation of M_1. We may suppose that S_2 contains at least two elements of the triangle $\{e, f, g)$ of M_1, say e and f. Then $\bar{S}_1 = S_1 - \{g\}$ and $\bar{S}_2 = S_2 \cup \{g\}$ make up a 3-separation $(\bar{S}_1, \bar{S}_2)$ of M_1, and $|\bar{S}_2| \geq 4$. If S_2 contains a base of M_1, then S_1, which contains at least four elements, has rank equal to 2; but then M_1 has two parallel elements since it is binary, a contradiction of the 3-connectivity of M_1. Thus S_2, and hence $\bar{S}_2$, cannot contain a base of M_1. Let $\bar{X}_2$ be a base of $\bar{S}_2$ containing e and f (and hence not g), then choose $\bar{X}_1 \subseteq \bar{S}_1$ to get a base $\bar{X}_1 \cup \bar{X}_2$ of M_1. The related representation matrix $\hat{\bar{B}}$ has $\bar{B}$ like the matrix of (5.3) with bars added to all symbols. Then $\bar{A}^1$ is nonempty since $\bar{X}_1$ is, and $\bar{A}^2$ has length of at least 4 since $\{e, f, g\} \subset \bar{S}_2$. We may assume that $\bar{A}^1$ has no zero column since this can always be achieved by shifting any such column from $\bar{Y}_1$ to $\bar{Y}_2$. Since $e, f \in \bar{X}_2$ and $g \in \bar{Y}_2$, we may suppose $\bar{B}$ to be also B^1 of (5.4) (if this is not so, pivots within A^1 of B^1 produce this case). Thus $\bar{A}^1$ is a proper submatrix of A^1, and has no zero column. Indeed, $\bar{A}^1$ must have length of at least 4 since otherwise a triad of M does not contain any element of $X_2 \cup Y_2$. Define $\tilde{X}_1 = \bar{X}_1$, $\tilde{X}_2 = (X_1 \cup X_2) - \tilde{X}_1$, $\tilde{Y}_1 = \bar{Y}_1$, $\tilde{Y}_2 = (Y_1 \cup Y_2) - \tilde{Y}_1$.

Repartition B according to these sets, and add tildes to all symbols. Existence of the desired 3-sum $\tilde{M}_1 \oplus_3 \tilde{M}_2$ then follows from Lemma 5.2. The claimed 3-connectivity of $\tilde{M}_1$ is assured by the fact that $\tilde{A}^1 (= \bar{A}^1)$ has no zero column; see Theorem 3.11 of [11]. □

Proof of Theorem 5.1. Suppose (1), (2), and (4) do not apply to M. By (4) of Theorem 4.3, $M = M_1 \oplus_3 M_2$ such that the related B of type (3.1) has $\bar{Y}_2 = \{l\}$ and M_1 is 3-connected. If M_1 is regular or $M_1 = F_7$, we are done. Otherwise M_1 has eight or more elements. Since $\bar{Y}_2 = \{l\}$, M_1 occurs in $\mathcal{M}$, so again by Theorem 4.3, M_1 has a $(3+)$-separation. By Lemma 5.5, $M = \tilde{M}_1 \oplus_3 \tilde{M}_2$ where $\tilde{M}_1$ is 3-connected, $\tilde{X}_1 \cup \tilde{Y}_1 \subset X_1 \cup Y_1$, and $l \in \tilde{Y}_2$. Furthermore by pivots in $\tilde{A}^2$ we can ensure that two 1's of column l reside in two rows where the related row submatrix of D has rank equal to 2. Thus $\tilde{M}_1$ is again in $\mathcal{M}$, and $\tilde{M}_1$ is $(3+)$-separable or regular or equal to F_7. By induction we are thus assured of (3). □

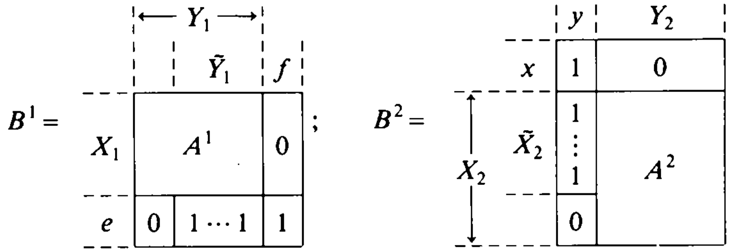
The proofs of Theorems 4.3 and 5.1 were purposely so chosen that they are useful for proof of the following claim.

Theorem 5.6. *There exists a polynomial algorithm to decide which case of Theorem 4.3 (Theorem 5.1) applies to a given connected (3-connected) $M \in \mathcal{M}$.*

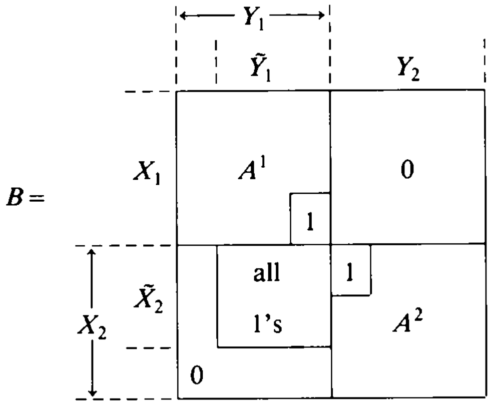
Proof. 2-separability of M is easily decided (see, e.g., [10]), so we may assume M to be 3-connected. Using Seymour's decomposition for regular matroids [6] one can construct several polynomial algorithms, each of which either declares M to be regular or $M = F_7$, or produces a minor N for (4) of Theorem 4.3 (The order of some of these algorithms is high, and we conjecture that faster methods can be constructed using some of the ideas of [10, 13]. Indeed, we conjecture that most fast algorithms for deciding regularity can be expanded to decide which case of Theorem 4.3 or 5.1 applies, without any increase in the order of the algorithm. This conjecture is motivated by the structural similarity of the two problems. For this reason we refrain here from citing any complexity bounds.) If M is regular or $M = F_7$, we stop, so suppose that an N minor has been produced. We then employ the partitioning algorithm of Section 3 at most three times, starting each time with another 3-separation of N as given by the table of (4) of Theorem 4.3, and testing whether that 3-separation induces one for M. Success is guaranteed for at least one of the cases, and the 3-separation of M so found is converted to a 3-sum satisfying (4) of Theorem 4.3 by row/column shifting as described in the proof of that theorem. Repeated application of the above procedure and of the operations described in the proof of Theorem 5.1 then gives a polynomial algorithm to select a case of Theorem 5.1 for M. □

Left open so far is the question of efficient testing of a binary M for membership in $\mathcal{M}$. For 2-separable matroids the following result is helpful.

Theorem 5.7. *Let M_1 and M_2 be connected binary matroids with representation matrices $\hat{B}^1$ and $\hat{B}^2$ respectively, where*

(5.8) 

Suppose an element l occurs only in $X_2 \cup Y_2$. Then $M = M_1 \oplus_2 M_2$, which is represented by $\hat{B}$ with

(5.9) 

is in $\mathcal{M}$ if and only if $\bar{M}_1, M_2 \in \mathcal{M}$, where $\bar{M}_1$ is derived from M_1 by relabelling f as l.

Proof. For proof of the 'if' part suppose M has a minor F_7^* such that $l \in F_7^*$. Let Z be the groundset of such a minor. If $Z \subseteq X_2 \cup Y_2$, then F_7^* is a minor of M_2, a contradiction. $Z \subseteq X_1 \cup Y_1$ is impossible, so by the 3-connectivity of F_7^*, $|Z \cap (X_i \cup Y_i)| = 1$ for $i = 1$ or 2. If $i = 1$, we may without loss of generality assume that $|Z \cap \tilde{Y}_1| = 1$, which proves that $M_2 \notin \mathcal{M}$. If $i = 2$, we similarly see that $\bar{M}_1 \notin \mathcal{M}$. The 'only if' part is even simpler. Clearly $M_2 \in \mathcal{M}$. Furthermore M is connected and has $\bar{M}_1$ as a minor, so $\bar{M}_1 \in \mathcal{M}$. □

Finding the analog of Theorem 5.7 for a 3-sum $M = M_1 \oplus_3 M_2$ is more difficult. Reference [7] contains a version using excluded minors that may possibly be of help in the design of a polynomial testing algorithm for membership in $\mathcal{M}$. More recently a quite different approach was pursued that finally did produce an analog of Theorem 5.7 for 3-sums of attractive (and quite unexpected) simplicity. The latter result rather easily leads to a polynomial testing algorithm for membership in $\mathcal{M}$. Details are described in [14].

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